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ABOUT QUASIPERIODIC MANIFOLDS AND CATEGORIES

Bordism relation of quasi-periodic manifolds is determined so that bordism invariant of the signature type would take non-trivial numerical values. The categories connected with asymptotic representation of groups are built.

Key words: *quasi-periodic, oriented, manifold, bordism, differential form, Morse function, partition, enlargement, signature, asymptotic.*

Investigation of geometric objects using algebraic methods has led to new views and theories in both geometry and algebra. The algebra of manifold functions is known to possess sufficiently complete information about a given manifold. In [4], a somewhat inverse approach to studying manifolds is proposed, and namely: not to associate manifolds with algebras but to associate algebras with manifolds, the function algebras of which are isomorphic to initial algebras. The notion of quasi-periodic oriented manifold generalizes the notion of periodic oriented manifold that is a universal covering of a compact oriented manifold, the fundamental group of which is isomorphic to an infinite cyclic group **Z**. The paper consists from two parts. In the first part bordism relation of quasi-periodic manifolds is determined so that the bordism invariant of a signature type would take non-trivial numerical values. The second part deals with building categories associated with asymptotic representations of a group.

Interest to quasi-periodic manifolds is connected with the fact that such manifolds are used in various problems of topology and geometry. For example, the problem of quasi-periodic manifold structure arises while considering level surfaces of multi-valued Morse functions given by Morse differential forms with incommensurable periods [1], [2], [3].

Basic definitions.

Periodic manifold can be described as such open manifold W_∞ that is represented as a combination of a calculable number of copies W_k, $-\infty < k < \infty$ of a compact oriented manifold V, $\varphi_k : \mathrm{W}_k \to \mathrm{V}$, the boundary of which consists of two non-intersecting copies of manifold M with opposite orientations: $\partial \mathrm{V} = M_+ \sqcup M_-$. Therefore, $\partial \mathrm{W}_k = M_k^- \sqcup M_k^+$, and integration $\mathrm{W}_\infty = \bigcup_k \mathrm{W}_k$ means identification of the points of the boundary $M_k^+ \cong M_{k+1}^-$. Naturally, an infinite group Z is acting freely on manifold W_∞ by shifting the points from one term to another, $a(x) = \varphi_{k+1}^{-1}\varphi_k(x)$, $x \in \mathrm{W}_\mathrm{k}$, $a \in \mathbf{Z}$ - a generating element. Under the action of the group the manifold factor is diffeomorphic to manifold V after identification of its points on the boundary because of identification $M^+ \cong M^-$.

In other words, a periodic (oriented) manifold or, more precisely, a periodic structure on a manifold is characterized by the following data set:

– *open (oriented) manifold* W_∞;

– *partition of* W_∞ *into submanifolds*;

$$W_\infty = \bigcup_{k=-\infty}^{\infty} W_k\ ; \tag{1}$$

– the system of homomorphisms (diffeomorphisms)

$$\varphi_k : \mathrm{W}_k \to \mathrm{V}, \tag{2}$$

where V – oriented compact manifold with the boundary, i.e. manifolds W_k are simulated by a certain (single) manifold V;

– covering conditions (1):

$$\partial \mathrm{W}_k = (\mathrm{W}_{k-1} \cap \mathrm{W}_k) \sqcup (\mathrm{W}_k \cap \mathrm{W}_{k+1}) = M_k^- \sqcup M_k^+, \tag{3}$$

$$\mathrm{W}_k \cap \mathrm{W}_{k+1} = \partial \mathrm{W}_k \cap \partial \mathrm{W}_{k+1} = M_k^+ = M_{k+1}^-. \tag{4}$$

Data set (1), (2), (3), (4) induces a free action of the cyclic infinite group $\mathbf{Z}$ on the manifold W_∞, transforming it to a total covering space above the compact manifold.

The structure of a quasi-periodic manifold on W_∞ is distinguished from the structure of a periodic

manifold only by the fact that homomorphisms (2) are replaced by diffeomorphisms

$$\varphi_k : \mathrm{W}_k \to \mathrm{V}_\alpha,\ \alpha = \alpha(k), \tag{5}$$

where index $\alpha \in A$ ranges over a finite set, i.e. manifolds W_k are simulated by several manifolds V_α, function $\alpha = \alpha(k)$ not being constant within any of the intervals $(-\infty, k]$ or $[k, \infty)$. The rest of the conditions (1), (3), (4) are preserved when the structure of a quasi-periodic manifold is determined.

Let us consider, for instance, a complex manifold X with fundamental group $\pi_1(X) = \mathbf{Z} \oplus \mathbf{Z}$. Let $\omega \in \Omega_1(X)$ be a close Morse form on manifold X. Assume that $p : \widetilde{X} \to X$ is a universal covering, $\widetilde{\omega} = p^*(\omega)$, f – Morse function on covering $\widetilde{X}$, $df = \widetilde{\omega}$. Let us consider a non-singular value c of function f and a level surface $\mathrm{W}_\infty = \{f = c\}$. Assume that $a, b \in \pi_1(X)$ are generatrixes of the fundamental group and the periods

$\lambda_1 = \int_a \omega$, $\lambda_2 = \int_b \omega$

are incommensurable. So they, at least, are not equal to zero. Let $Y \subset X$ be a submanifold of a codimension 1 dual, for instance, to element $a \in H_1(X; \mathbf{Z})$. Without losing generality, we can assume that manifolds W_∞ and $p^{-1}(Y)$ intersect transversally and restriction of function f on submanifold $p^{-1}(Y)$ is a function of Morse. Then submanifold $p^{-1}(Y)$ divides manifold W_∞ into the component that form the structure of a quasi-periodic manifold.

The structure of a quasi-periodic manifold could be subjected to two natural operations: *partition* and *enlargement.*

Partition means that manifolds V_α, can, in their turn, be represented in the form of a finite division of the type (1), (5), (3), (4), which makes it possible to obtain a smaller division of manifold W_∞.

Enlargement means that according to a strictly given monotonic sequence n_k, under the condition $n_{k+1} - n_k < Const$, manifolds $W_k' = W_{[n_k \ldots n_{k+1}-1]}$ are taken as the division.

Let F be a left-invariant normalized functional on the space $L^\infty(\mathbf{Z})$ of restricted functions on a free cyclic group $\mathbf{Z}$. Let us consider function $s(m), m \in \mathbf{Z}$ given by the formula

$$s(m) = \frac{\mathrm{sign} X_{[k_{n+1} \cdots l_{n+1}]} - \mathrm{sign} X_{[k_n \cdots l_n]}}{l_{n+1} - k_{n+1} - l_n + k_n - 2},\ k_{n+1} \le m \le k_n \text{ или } k_n \le m \le k_{n+1}.$$

Theorem 1. *Function* $s(m)$ *is limited, i.e.* $s \in L^{\infty}(\mathbf{Z})$ *and value of the functional*

$$sign(W_{\infty}) = F(s)$$

does not depend on the choice of sequences k_n, l_n.

Theorem 1 requires additional study of the invariance of $\mathrm{sign}(\mathrm{W}_{\infty})$ value, specified in the theorem, with respect to a certain bordism relation of quasi-periodic manifolds. Let there be given a quasi-periodic (oriented) manifold W_{∞}, i.e. the structure of a quasi-periodic manifold in the form of partition (1), satisfying the conditions (5), (3), (4). We say that manifold W_{∞} is *bordant to zero* if it is the boundary of another (oriented)) manifold P_{∞} that allows partition in the form of the unification

$$\mathrm{P}_{\infty} = \bigcup_{k=-\infty}^{\infty} \mathrm{P}_k ,$$

of homomorphisms

$$\psi_k : P_k \to Q_{\alpha}, \ \alpha = \alpha(k),$$

satisfying the conditions

$$\partial P_k = (P_{k-1} \cap P_k \sqcup \mathrm{W}_k \sqcup (P_k \cap P_{k+1}) = N_k^{-} \sqcup \mathrm{W}_k \sqcup N_k^{+} ,$$

$$P_k \cap P_{k+1} = \partial P_k \cap \partial P_{k+1} = N_k^{+} = N_{k+1}^{-} ,$$

$$\partial N_k^{+} = M_k^{+} = M_{k+1}^{-} = \partial N_{k+1}^{-} .$$

Orientations of all manifolds should be coordinated according to the usual rules.

Conditions (3), (4), (5) define the structure of the so called quasi-periodic manifold on the manifold W_{∞}. We can determine an appropriate equivalence relation of such structures. In other words, by defining the class of the so called quasi-periodic maps, two structures of the quasi-periodic manifolds will be considered equivalent if the identity mapping from one structure to the other is a quasi-periodic map. In particular, let us consider a quasi-periodic structure

$$W_{\infty} = \bigcup_{k=-\infty}^{\infty} W_k$$

and a certain enlargement of this structure

$$W_{\infty} = \bigcup_{k=-\infty}^{\infty} W'_k = \bigcup_{k=-\infty}^{+\infty} W_{[n_k \ldots n_{k+1} - 1]}$$

with the condition

$$1 \le n_{k+1} - n_k \le \lambda = Const . \tag{6}$$

Condition (6) means that

$$k \le n_k \le \lambda k .$$

Thus, let there be given two quasi-periodic manifolds W_{∞} and W'_{∞} with the structure of quasi-periodic manifolds described by the conditions (1), (5), (3), (4). Map $f : W_{\infty} \to W'_{\infty}$ is referred to as *quasi-periodic* if there is a sequence n_k and a constant λ that satisfy the conditions:

1. $1 \le n_{k+1} - n_k \le \lambda = Const$;
2. $f(W_k) \subset W'_{[n_k \ldots n_{k+1} + \lambda]}$;

3. *Maps*

$$u_k = (\varphi_k \cup \varphi_{k+1}) \circ f \circ \varphi_k^{-1} : V_{\alpha(k)} \to \left(\bigcup_{j=n_k}^{n_{k+1}+\lambda} V'_{\alpha'(f)} \right)$$

take a finite number of values.

Theorem 2. *The identity mapping of a quasi-periodic structure into its enlargement is a quasi-periodic map.*

The composition of quasi-periodic maps is a quasi-periodic map.

If a quasi-periodic map is a diffeomorphism, then the inverse map is also a diffeomorphism.

Theorem 2 introduces the equivalence relation on the set of all quasi-periodic structures. The problem arises of describing the set of equivalency classes of quasi-periodic structures. We make a hypothesis that in any quasi-periodic manifold there is only one class of equivalent quasi-periodic structures. This hypothesis is based on the following consideration. If manifold W_∞ is divided into the following quasi-periodic structures in two ways

$$\mathrm{W}_\infty = \bigcup_{k=-\infty}^{\infty} \mathrm{W}_k = \bigcup_{k=-\infty}^{\infty} \mathrm{W}'_k ,$$

then each submanifold W'_k lies in the finite union $\mathrm{W}'_k \subset \bigcup_{j=n_k}^{m(k)} W_j$. In our hypothesis there is an essential consideration that $m(k) - n(k)$ must be restricted by a certain constant. Otherwise, manifolds W'_k cannot be diffeomorphic to one another from purely homological considerations .

Categories of asymptotic representations

Let us build two categories related to asymptotic representations. The first of them is built in the following way. Let $A - C^*$ be algebra with unit, G – a topologic (compact) group. We denote the class of all asymptotic representations of group G as $a.R(G, A)$ in Hilbert $C^* - A$ modules. Further, M and L are Hilbert modules above algebra A and $a.G$ - given representations of $\Phi = \{\Phi_n : G \to GL(M)\}_{n\in N}$ and $\Psi = \{\Psi_n : G \to GL(L)\}_{n\in N}$ of group G in modules M and L respectively. Each $a.G$ representation Φ of group G in module M defines in module M $a.G$ an action, and namely: family $\varphi = \{\varphi_n\}_{n\in N}$, determined by formula $\varphi_n(g,m) = \Phi_n(g)(m)$, is action $a.$ of group G in module M . Therefore, (M,φ) is $a.G$ module. In the same way $a.G$ module (L,ψ) is determined. Let us denote the class of all A homeomorphisms as $a.Hom_A(M,L)$ $f : M \to L$, that satisfy the condition **C1** of [3]. The same homeomorphisms we will call intertwining homomorphisms $a.$ of representations Φ, Ψ and denote this as $f : \Phi \to \Psi$. The set of all intertwining homomorphisms Φ and Ψ we denote as $mor(\Phi,\Psi)$. Hence,

$$mor(\Phi,\Psi) = a.Hom_A(M,L). \tag{7}$$

It is easy to verify the validity of the following properties .

Property 1. The sum of intertwining homomorphisms is also an intertwining homomorphism, i.e., if $f, g \in mor(\Phi,\Psi)$, then $f + g \in mor(\Phi,\Psi)$.

Property 2. If $f : \Phi \to \Psi$ is an intertwining homomorphism and $a \in A$, then af is also an intertwining homomorphism.

Property 3. Intertwining homomorphisms form a submodule $mor_A(\Phi,\Psi)$ in A module $Hom_A(M,L)$.

The set of all intertwining homomorphisms of $a.$ representations of group G we denote as $a.Mor(G)$. The pair of classes $(a.R(A,G), a.MorG)$ form a category of asymptotic representations and intertwining homomorphisms. Let us denote this category as **a.R** $(A,G)_1$.

The second category of asymptotic representations is built in the following way. The class of objects $a.R(A,G)$ does not change. Let $\Phi = \{\Phi_n : G \to GL(M)\}_{n\in N}$ and $\Psi = \{\Psi_n : G \to GL(L)\}_{n\in N}$ be two $a.G$ representations from $a.R(A,G)$. Family $\alpha = \{\alpha_n : M \to L\}_{n\in N}$ of A-homomorphisms we will call asymptotically intertwining ($a.$-intertwining) homomorphisms $a.G$ of representations Φ and Ψ, if the following condition is satisfied:

$$\|\alpha_n \circ \Phi_n(g) - \Psi_n(g) \circ \alpha_n\| \to 0 \text{ for } n \to \infty. \tag{8}$$

If α is $a.$-intertwining homomorphism of $a.$-representations of Φ и Ψ, then we denote this as $\alpha : \Phi \sim \Psi$.

Statement 1. The relation of asymptotic intertwinity is reflexive and transitive.

Comparing the class of morphisms of each category with a certain groupoid, we can interpret the homomorhisms of groupoids as functors. Let $(\mathbf{G}, \mathbf{G}'; \varphi)$ and $(\mathbf{H}, \mathbf{H}'; \varphi)$ be groupoids and $K_{\mathbf{G}}$ and $K_{\mathbf{H}}$ – the corresponding categories, and namely: correspondences $\mathsf{E}_{\mathbf{G}} : \mathbf{G} \to MOR(K_{\mathbf{G}})$ and $\mathsf{E}_{\mathbf{H}} : \mathbf{H} \to MOR(K_{\mathbf{H}})$ are compared with groupoids $(\mathbf{G}, \mathbf{G}'; \varphi)$, $(\mathbf{H}, \mathbf{H}'; \varphi)$ and classes of morphisms of categories $K_{\mathbf{G}}$ and $K_{\mathbf{H}}$ respectively. Let us consider a contravariant functor $F : K_{\mathbf{G}} \to K_{\mathbf{H}}$. Homorphism $F^{\#} : \mathbf{G} \to \mathbf{H}$ we define in the following way: *if* $F(\mathbf{g}) = \mathbf{h}$, *then* $F^{\#}(g) = h$, *where* $\mathsf{E}_{\mathbf{G}} : (g) = \mathbf{g}$ *u* $\mathsf{E}_{\mathbf{G}} : (h) = \mathbf{h}$. Then for $\varphi(\mathbf{g}_1, \mathbf{g}_1) \in \mathbf{G}'$ we have $F(\varphi(\mathbf{g}_1, \mathbf{g}_1)) = \psi(F(\mathbf{g}_1), \mathrm{F}(\mathbf{g}_1))$.

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