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TRIGONOMETRIC REPRESENTATION OF CUBIC HERMITIAN SPLINES

This paper proposes a new trigonometric presentation of cubic Hermiian splines, which makes it possible to obtain more accurate interpolation results compared with the existing spline methods. Such results are achieved for almost periodic, quasiperiodic functions and vibration resonance functions. Brief overview of spline interpolation and trigonometric interpolation methods is presented. The benefits and key application areas of the proposed trigonometric interpolation method are analyzed. The paper presents errors in the results of test function interpolation using the developed and the existing spline interpolation methods.

Key words: *interpolation, spline, trigonometric interpolation, almost periodic, quasiperiodic, vibration resonance.*

Introduction

Continuous hardware development in modern informational world requires creation and implementation of new effective methods and models of data processing and visualization. This applies to all branches of modern science. This work aims to improve and extend the application of spline interpolation. The paper proposes a new method of trigonometric spline interpolation for discrete data processing. The proposed method provides more accurate results for almost periodic and quasiperiodic functions as compared with similar methods.

One of the interesting phenomena of mathematics and theoretical physics, the investigation of which started quite recently, is vibration resonance (VR). VR is quasi-periodic motion (function) presented as a change of a dynamic system and is characterized by integration of a finite number (two or more) of incompatible signals. Incompatible signals are understood as periodic signals that differ radically in their amplitude and frequency. VR can be observed in a nonlinear system when two different periodic signals (a strong and a weak one) are applied to the system input. In this case a weak low-frequency signal can be amplified with a powerful high-frequency one. VR effect is similar to stochastic resonance (SR), where the noise is replaced by a high-frequency signal [1]. It should be noted that signals with two radically different frequencies can be found in such fields as communications, acoustics [2], neurobiology [3] and physics of laser [4]. Therefore the task of accurate processing and representation of periodic and quasi-periodic functions is of current importance [5 – 8].

Overview of modern spline interpolation methods

Interpolation is the process of calculating the intermediate values of an unknown function given by the grid of discrete values. Spline interpolation is one of the interpolation methods that represents an unknown function between the neighboring points of the grid as a polynomial of an integer power [9]. This section provides a brief review of modern spline interpolation methods in order to emphasize their importance in the theory of discrete data processing and for better understanding of the modern trends of spline interpolation theory development. Modern types of spline interpolation can be classified into the following groups:

Auto-Tensioning splines

Splines with uniform tension were introduced by Schweikert in1966. In 1987 Renka proposed interpolation with the application of hyperbolic functions. For each interval between the given neighboring points minimal tension factor is calculated, for which the conditions of continuity of the first and the second derivatives in the nodal point of interpolation are observed [10].

Hermitian splines

On each segment the resultant curve is a third-degree polynomial constructed on the basis of the given slope angles (the derivatives) in the nodal points and the conditions of the polynomial passing through the nodal points. Each interval is normalized to the variable $t = \frac{x - x_i}{x_{i+1} - x_i}$, $t \in [0,1]$. Hermitian spline polynomial for the segment $[x_i,\ x_{i+1}]$ will be given by

$$S_i(t) = (2t^3 - 3t^2 + 1)y_i + (t^3 - 2t^2 + t)m_i + (-2t^3 + 3t^2)y_{i+1} + (t^3 - t^2)m_{i+1},$$

where m_i, m_{i+1} are values of the derivatives in the nodal points (x_i, y_i) and (x_{i+1}, y_{i+1}) respectively. For calculating the values of m_i both accurate and approximate methods can be used (Catmull-Rom spline).

B-splines (Basic splines)

B-splines were developed and published in the 1970s on the basis of Bezier splines. Bezier splines are given by the expression

$$P_o f_0(t) + P_1 f_1(t) + P_2 f_2(t) + P_3 f_3(t),$$

where $f_i(t), i = \overline{0;3}$ are transition functions that determine the degree of the point influence on the resultant curve. In the case of Bezier curves the Bernstein polynomial is used [11]. P_0, P_3 are the first and last points of the curve. P_1, P_2 are the points which determine the form of the resultant curve but do not belong to it.

NURBS (Non-Uniform Rational Basis Splines)

One of the advantages of NURBS is their ability to provide accurate presentation of both curves and conic surfaces [12]. NURBS provide a possibility to present both curves and surfaces taking into account their form, physical (geometrical) or parametric (mathematical) continuity requirements. NRBS curve is constructed not only on the basis of the nodal points but also taking into account weight coefficients determined for each nodal point. General NURBS formula is written as

$$C(u) = \frac{\sum_{i=0}^{n} \omega_i P_i N_{i,k(u)}}{\sum_{i=0}^{n} \omega_i N_{i,k(u)}},$$

where ω_i are weight coefficients, P_i are the nodal points, $N_{i,k}$ are the normalized basic spline functions of k power.

Splines of the fifth order (Quintic Splines)

They are created for constructing a smooth curve that is stable to perturbations of the basic points. In this type of spline interpolation the first four derivatives are continuous in each interval (x_i, x_{i+1}). For each interpolation interval the spline function is given by the expression

$$S(x) = y_i + B_i t + C_i t^2 + D_i t^3 + E_i t^4 + F_i t^5,$$

where $t = x - x_i$, $x_i \le x < x_{i+1}$, $i = \overline{1, N}$. B_i, C_i, D_i, E_i, F_i are spline coefficients [13].

X-Spline

Blanc and Schlick proposed an entirely new approach to spline function construction [14]. The idea is that each nodal point P_k influences the four spline segments of the resultant curve and so smoothing function F_k is non-zero in the four sequential intervals (F_k becomes non-zero in the node t_{k-2}, maximize in t_k and becomes zero in the node t_{k+2}). Blank and Schlick derived the following formula of X splines for the interval $[t_{k+1}, t_{k+2}]$ using basic points $P_k, P_{k+1}, P_{k+2}, P_{k+3}$:

$$C(t) = \frac{A_0(t)P_k + A_1(t)P_{k+1} + A_2(t)P_{k+2} + A_3(t)P_{k+3}}{A_0(t) + A_1(t) + A_2(t) + A_3(t)}, \qquad (1)$$

where $A_0(t), A_1(t), A_2(t), A_3(t)$ are the coefficients obtained from the conditions of intersection of two smoothing functions F_{k-2} and F_{k+2}. The above X-Spline formular equation (1) allows performing approximation (the resultant curve does not pass through the given points). Blanc and Schlick introduce an additional parameter $s_k, k \in [0;1]$. In case, $s_k = 0$ the curve passes exactly through P_k points .

Overview of the existing analogs of trigonometric interpolation methods

Since almost all periodic and quasiperiodic functions are based on the sum of harmonic signals, it is natural to pass from polynomial to trigonometric spline representation form. In publications connected with interpolation trigonometric functions are often used for spline representation. Let us analyze the most interesting and weighty among them.

One of the most common (classic) trigonometric interpolation methods uses function decomposition into Fourier series [15]. Periodic function $f(t) = f(t+T), \forall t \in (-\infty;\infty)$ using Fourier series decomposition will be written as:

$$f(t) \approx p_n(t) = a_0 + \sum_{j=1}^{n} (a_j \cos(j \cdot t) + b_j \sin(j \cdot t)), \; |a_n| + |b_n| \neq 0,$$

where n is a number of terms in Fourier series for the spline interpolation formula, T is a period. Let us assume the function period to be $T = 2\pi$ (this value can be changed easily by introduction of an additional coefficient). Then for the entire interpolation function description it is sufficient to set the grid of the points:

$$f(t_i) = p_n(t_i); \; i = 0, 1, 2, \ldots, 2n; \; 0 \le t_0 \le t_1 \le t_2 \ldots \le t_{2n} \le 2\pi.$$

Using the above-mentioned initial data, the values of unknown coefficients a_j and b_j are calculated by the formulas:

$$a_j = \frac{2}{2n+1} \sum_{k=0}^{2n} f(t_k) \cos(j \cdot t_k); \; b_j = \frac{2}{2n+1} \sum_{k=0}^{2n} f(t_k) \sin(j \cdot t_k).$$

Periodic functions are the main area of application of trigonometric splines based on Fourier series. This interpolation method application for pseudoperiodic or non-periodic functions is not feasible due to its low accuracy as compared with the application of cubic splines or Hermitian splines.

Another interesting form of spline interpolation representation using trigonometric functions is given in [16]. This interpolation method is given by

$$S_i(t) = y_i \left[1 - \frac{1 - \cos(\pi \cdot t)}{2} \right] + y_{i+1} \left[\frac{1 - \cos(\pi \cdot t)}{2} \right], \; t = \frac{x - x_i}{x_{i+1} - x_i}, t \in [0;1], \qquad (2)$$

where $S_i(t)$ is interpolation polynomial of the function in the interval $[x_i; x_{i+1}]$; t is an additional normalized variable.

Even though spline form (2) is very simple, it has one major drawback: the value of the first derivative of the interpolated function in the main grid points is zero. Such a drawback restricts application of this trigonometric form of spline representation in practical tasks.

In [17] Robert F. Kauffmann proposed a trigonometric spline calculation on the basis of four basic points. Using matrix representation, trigonometric spline of Robert Kauffmann will be written as

$$F = \frac{1}{2}\begin{bmatrix} \cos(\pi \cdot t/2)(\cos(\pi \cdot t/2) - 1) \\ \sin(\pi \cdot t/2)(\sin(\pi \cdot t/2) + 1) \\ \cos(\pi \cdot t/2)(\cos(\pi \cdot t/2) + 1) \\ \sin(\pi \cdot t/2)(\sin(\pi \cdot t/2) - 1) \end{bmatrix}; P_x = \begin{bmatrix} x_i \\ x_{i+1} \\ x_{i+2} \\ x_{i+3} \end{bmatrix}; P_y = \begin{bmatrix} y_i \\ y_{i+1} \\ y_{i+2} \\ y_{i+3} \end{bmatrix}; t \in [0;1]; \qquad (3)$$

$$S_x(t) = P_x^T \cdot F; \; S_y(t) = P_y^T \cdot F.$$

Artificial variable $t \in [0;1]$ is introduced into expression (3). Apart from calculating the value of the interpolation function $S_y(t)$, $S_x(t)$, argument is also calculated using four points of the grid of the given values P_x and P_y.

This type of spline interpolation is better to be used for building circles or other conical figures.. To the disadvantages of this spline belong: low interpolation characteristics as compared with Hermitian or cubic splines; complicated calculation of the interpolation function value in the given point; the impossibility to perform interpolation in the first and the last intervals of the grid of set points.

Paper [18] proposes interpolation with the application of modified Hermitian splines. I The paper proposes replacement of the Hermitian spline polynomial by the empirically derived trigonometric expressions (table 1).

$$S_i(t) = f1(t) \cdot y_i + f2(t) \cdot y_i' + f3(t) \cdot y_{i+1} + f4(t) \cdot y_{i+1}'.$$

The proposed trigonometric spline variant enables more accurate interpolation function construction for certain pseudoperiodic functions as compared with cubic Hermitian splines as well as setting initial boundary conditions to the values of high-order derivatives. Disadvantages of this method are empiric deduction of the expressions for basic functions and impossibility to use such splines in general practical tasks.

The presented overview of trigonometric spline interpolation models reveals their general shortcomings – their narrow specialization and, in some cases, their complicated algotithmization.

Table 1

Comparison of the expressions for basic functions of Hermitian and trigonometric splines

Original basic functions of Hermitian cubic spline	The proposed trigonometric expressions for basic functions
$f1(t) = 2t^3 - 3t^2 + 1;$ $f2(t) = t^3 - 2t^2 + t;$ $f3(t) = -2t^3 + 3t^2;$ $f4(t) = t^3 - t^2;$ $t = \frac{x - x_i}{x_{i+1} - x_i}; \; t \in [0,1].$	$f1(t) = \cos(t)^2;$ $f2(t) = \sin(t)^2;$ $f3(t) = 0.096225(\sin(3t) + \sin(t));$ $f4(t) = 0.096225(\cos(3t) - \cos(t));$ $t = \frac{x - x_i}{x_{i+1} - x_i}\frac{\pi}{2}; \; t \in [0, \frac{\pi}{2}].$

Main goal of the research is to develop a new model of the trigonometric spline interpolation that will make it possible to obtain more accurate results as compared with other methods for almost periodic and quasiperiodic functions without complicating the computation.

A modified trigonometric spline model

Let us assume that values of $f(x)$ function are known in the given points $f(x_i) = y_i$, $i = \overline{1,n}$, as well as values of the first derivatives in the extreme points $f'(x_1) = R_1$, $f'(x_n) = R_n$. In order to construct a modified trigonometric spline we take the following expression:

$$S(t) = a + bt + c \cdot \cos(t) + d \cdot \sin(t),\ t = \frac{x - x_i}{x_{i+1} - x_i} \cdot \frac{\pi}{2},\ t \in [0; \pi/2].$$

Variable x between the interpolation grid points $[x_i; x_{i+1}]$ is normalized to t variable. Let us find unknown coefficients a, b, c, d expressing them through the values of the function and its derivatives R_i and R_{i+1} in the interpolation interval points x_i and x_{i+1} (fig. 1).

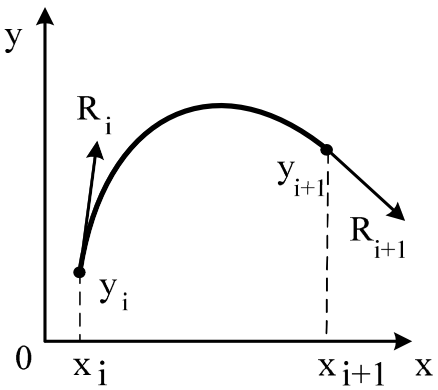


Fig.1. Trigonometric spline model construction

Let us calculate $S(t)$ and $S'(t) = b - c \cdot \sin(t) + d \cdot \cos(t)$ for $t = 0;\ t = \pi/2$.

$$S(0) = a + c; \qquad S'(0) = b + d;$$
$$S(\pi/2) = a + b \cdot \pi/2 + d;\ S'(\pi/2) = b - c.$$

Let us write the above expression in the matrix form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & \pi/2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} S(0) \\ S(\pi/2) \\ S'(0) \\ S'(\pi/2) \end{bmatrix} \text{ or } A \cdot K = S.$$

The previous equation is transformed to the form $A^{-1} \cdot S = K$. Unknown coefficients a, b, c, d in the matrix form can be expressed as

$$\frac{1}{(\pi - 4)} \cdot \begin{bmatrix} \pi - 2 & -2 & 2 & \pi - 2 \\ -2 & 2 & -2 & -2 \\ -2 & 2 & -2 & -(\pi - 2) \\ 2 & -2 & \pi - 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} S(0) \\ S(\pi/2) \\ S'(0) \\ S'(\pi/2) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Let us substitute the obtained values a, b, c, d into the general expression of spline $S(t)$:

$$S_i(t) = \frac{-2t + 2\sin(t) + (\pi - 2) - 2\cos(t)}{\pi - 4} S(t_i) + \\ + \frac{-2 + 2t + 2\cos(t) - 2\sin(t)}{\pi - 4} S(t_{i+1}) + \\ + \frac{-2\cos(t) + 2 - 2t + \sin(t)\cdot(\pi - 2)}{\pi - 4} S'(t_i) + \\ + \frac{-2t + (\pi - 2) - \cos(t)\cdot(\pi - 2) + 2\sin(t)}{\pi - 4} S'(t_{i+1}). \quad (4)$$

The obtained expression (4) is a modified trigonometric spline. Let us consider its features. We construct plots for the expressions that are coefficients $S(t_i)$, $S(t_{i+1})$, $S'(t_i)$ and $S'(t_{i+1})$ for the normalized variation range of parameter $t \in [0; \pi/2]$ (fig. 2). Plots of the basic functions $f1(t)$, $f2(t)$, $f3(t)$ and $f4(t)$ are similar to the plots of the basic functions which are used for Hermitian cubic splines.

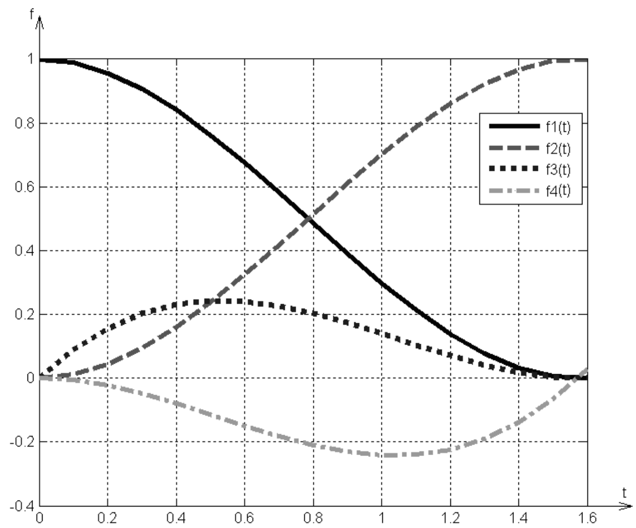


Fig.2. Plots of the basic functions of a modified trigonometric spline

Trigonometric functions $\sin(x)$ and $\cos(x)$ are used instead of a cubic polynomial. The required form of the resultant interpolation curve is obtained by the combination of basic functions and different coefficients.

In the modified trigonometric splines the equality condition for the first derivatives of the neighboring splines in the basic points of the interpolation grid is satisfied automatically.

For practical verification of the developed splines an approximate and precise methods for calculating the values of the function derivatives in the basic points were used. In the approximate method the following formula was used for estimating the derivative in the point

$$S'(t_i) = \frac{1}{2}\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} + \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right);\ i = \overline{2; n-1};\ \ S'(t_1) = R_1;\ S'(t_n) = R_n.$$

To calculate the derivatives using the analytic (precise) method, equality condition of the second derivative of neighboring splines in the basic grid point. As a result, we obtain a tridiagonal matrix for which simple, fast and effective solution methods are elaborated.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \pi-2 & 4 & \pi-2 & 0 & \dots & 0 \\ 0 & \pi-2 & 4 & \pi-2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \pi-2 & 4 & \pi-2 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_1' \\ S_2' \\ S_3' \\ \dots \\ S_{n-1}' \\ S_n' \end{bmatrix} = \begin{bmatrix} R_1 \\ 2S(t_3)-2S(t_1) \\ 2S(t_4)-2S(t_2) \\ \dots \\ 2S(t_n)-2S(t_{n-2}) \\ R_n \end{bmatrix}, \qquad (5)$$

where R_1, R_n are boundary conditions, the value of the first derivative of the function in the first and the last interpolation grid points respectively.

Verification of the developed modified trigonometric splines

In order to test and compare the results achieved with other existing interpolation methods, the proposed splines were realized in mathematical programming package Matlab. In order to calculate the derivatives $S'(t_i)$ and $S'(t_{i+1})$ in (4), we solve the equation set (5) obtained from the conditions of continuity of the first derivative in the common points of two neighboring splines.

The paper presents the comparative study of the results of test functions interpolation (table 2) by the developed trigonometric splines, cubic Hermitian and cubic splines. Average deviation from the preset function is chosen as a criterion of accuracy. For testing the spline methods the following function types were chosen:

1. Polynomials;
2. Trigonometric functions;
3. Fractional rational functions;
4. Quasiperiodic functions;
5. Vibration resonance quasiperiodic functions.

Apart from vibration resonance function, the paper considers one of the quasiperiodic functions – θ function for testing the developed splines. This is an analytic function of two variables $\theta(z,\tau)$ that is given by the series:

$$\theta(z,\tau) = \sum_{n \in Z} \exp(\pi i n^2 t + 2\pi i n z),$$

where $z \in C$ and $\tau \in H$ (H - an upper half-plane $\operatorname{Im}(\tau) > 0$). As it is shown in [19] this series converges absolutely inn any compact subset $C \times H$. Quasiperiodic feature of θ function is manifested in shifting $z \to z+\tau$:

$$\theta(z+\tau,\tau) = \exp(-\pi i\tau - 2\pi i z)\theta(z,\tau),$$

i.e. function θ is quasiperiodic with respect to $\Lambda_\tau \in C$ based on 1 and τ. While shifting by an arbitrary count element, function θ is given by the formula:

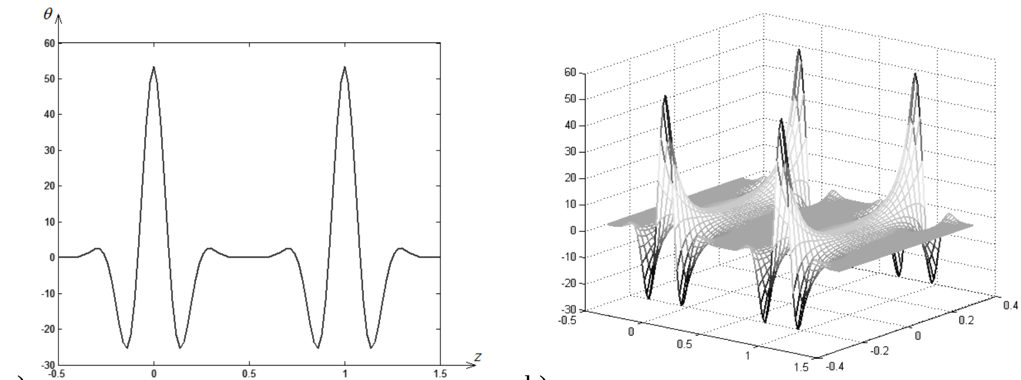
$$\theta(z + a\tau + b,\tau) = \exp(-\pi i a^2\tau - 2\pi i a z)\theta(z,\tau).$$

Table 2

Results of the test function spline interpolation using three methods. The table gives average deviations from the test functions. The marked cells show the best interpolation result

#	Test function	Range [-4, 4] Base point step: 0.8			Range [-1, 4] Base point step: 1			Range [-1, 8] Base point step: 1.5		
		Trigonomic spline	Hermit spline	Cubic spline	Trigonomic spline	Hermit spline	Cubic spline	Trigonomic spline	Hermit spline	Cubic spline
1	X*X-1;	0.2046	0.2503	0.0000	0.1879	0.1785	0.0000	0.0423	0.4834	0.0000
2	0.5*X*X-2*X+5;	0.1023	0.1267	0.0000	0.0940	0.0892	0.0000	0.0197	0.2304	0.0000
3	X^3+X^2+X-1;	1.2622	0.8164	0.0000	0.4503	1.0809	0.0000	1.0793	5.6380	0.0000
4	sin(X);	0.0166	0.0547	0.0004	0.0213	0.0807	0.0017	0.0015	0.1306	0.0164
5	sin(X-5)/(X-5);	0.0052	0.0133	0.0001	0.0085	0.0200	0.0006	0.2106	0.1919	0.2512
6	5*sin(2*(X-5))/(2*(X-5));	0.0088	0.0859	0.0210	0.0757	0.1275	0.0983	1.1452	1.2725	1.3286
7	cos(X + pi*sin(X));	0.0910	0.0767	0.0978	0.1812	0.1986	0.3667	0.5816	0.4872	0.5845
8	X*X+6*X*sin(2.35*X)-X;	1.2789	2.1681	0.9252	0.8722	2.6891	1.8888	7.8886	10.6902	16.0174
9	exp(-X)*X-4*sin(X)+X*fX	3.6537	1.7549	0.1794	0.2180	0.5029	0.0251	0.0621	0.7292	0.1060
10	X^3+X^2+X-1-0.1*X^4;	1.2622	0.8017	0.0054	0.6073	0.7448	0.0244	0.0791	1.9092	0.0975
11	Theta(X);	20.2603	16.5029	17.2130	51.6280	51.9296	51.9296	30.4656	29.8112	26.3949
12	X-X^3+cos(0.3*X)+10*cos(3*X);	1.6702	2.5697	2.2968	1.3421	1.7829	3.5551	7.4312	7.8747	8.3363

In order to test the developed trigonometric splines, one-dimensional function θ . shown in fig.3a, was used.



a) b)

Fig.3. Two-dimensional a) and three-dimensional b) plots of function θ

Conclusions

A new method of trigonometric spline interpolation is elaborated. Unlike the existing methods, it makes it possible to improve interpolation accuracy for almost periodic discrete data processing. Hermitian cubic spline construction model was used in the process of this new method creation in order to reduce the number of unknown coefficients.

The proposed one-dimensional trigonometric spline interpolation model was extended for two-dimensional discrete data processing. To calculate unknown spline coefficients, approximate and analytical approaches are proposed.

Program realization of the proposed method in the Matlab mathematical software package has been performed. The paper presents the results of test function interpolation with the application of both known and the elaborated method. As a result of the obtained data processing, a conclusion was made that the proposed method of trigonometric spline interpolation has better interpolation characteristics for almost periodic functions as compared with other methods.

Further research and development of the problem of almost periodic function processing must proceed in the direction of increasing dimensionality (the number of independent variables) and extending the number of practical tasks of the elaborated model application.

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