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STUDYING CONVERGENCE OF THE FINITE STRIP METHOD IN SOLVING PLANE ELASTICITY PROBLEM

The paper presents an algorithm of solving the plane elasticity problem by finite strip method and demonstrates its distinguishing features and advantages. Analysis of the results has been performed as compared with those obtained by the finite element method.

Keywords: *finite strip, finite strip method, finite element method, plane stress state.*

State of the art and research tasks

The finite strip method (FSM) has proved to be the most efficient one for solving certain plane elasticity problems, which deal with structures having constant physical-geometrical characteristics and simple boundary conditions. This is especially true for the problems where one size is much bigger than the other one. In such problems some simplifications could be introduced using general approximation functions. Usage of approximation functions is common both for finite element method (FEM) and for FSM, the difference being in that FSM uses only trigonometric functions for solving continual problem in longitudinal direction without discretization into simple functions. In opposite direction a discrete problem, created by division into finite strips separated by nodal lines, is solved. Their work is described by linear shape functions by analogy with FEM (Fig. 1, a).

A detailed bibliographical overview of FSM development is presented in paper [3]. A comprehensive description of FSM is given in monographs [1, 2].

The paper aims at studying convergence of the plane elasticity problem by the example of a console plate calculation by the finite strip method.

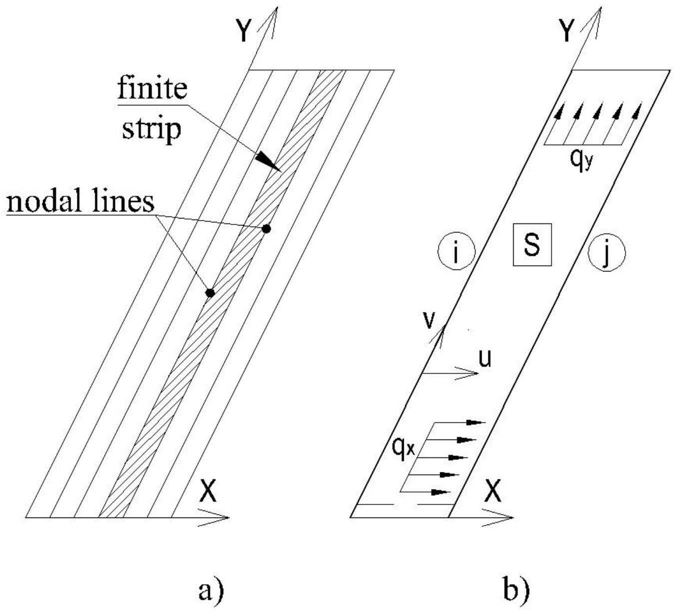

Fig. 1. The finite strip method a) finite-element strip; b)finite strip

Presentation of the main material

A thin isotropic plate with a uniform thickness is considered. To one end of it ligatures are applied, preventing any displacements in x. y-plane, while the other end moves freely. The load is assumed to be distributed uniformly per unit volume of the material and to be acting in the direction corresponding to the direction of the displacement vector $0Y$. Stress-strain state of the plate is described by the plane elasticity problem and is found using FSM in the displacements, i.e. an equivalent principle of the minimum potential energy is used, which is expressed through the field of possible displacements of the given shape. Let us consider a finite strip S, the behavior of which is described by the displacements of nodal lines *i* and *j* (Fig. 1, b).

Components of displacements *u*, *v* of finite strip *S* are approximated by functions in the form of $u(x, y)$ and $v(x, y)$ [1]:

$$u = \left[\left(1 - \frac{x}{b}\right) \quad \left(\frac{x}{b}\right)\right] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \sum_{m=1}^{r} Y_m(y)$$
$$v = \left[\left(1 - \frac{x}{b}\right) \quad \left(\frac{x}{b}\right)\right] \begin{Bmatrix} v_i \\ v_j \end{Bmatrix} \sum_{m=1}^{r} Y_m^{/}(y) \frac{L}{\mu_m}, \quad (1)$$

where r – a number of terms in the series; L – the finite strip length; b – the finite strip width, i, j – respective indices of nodal lines, $\mu_m = m\pi$.

$$Y_m(y) = \sum_{m=1}^{r} \left[1 - \cos\left(\frac{(m-0.5)\pi y}{L}\right)\right]. \quad (2)$$

It should be noted that through the choice of shape functions support conditions at the ends are provided as well as continuity conditions at the finite strip boundaries.

Functions (1) are presented in the form of

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{d\}, \quad (3)$$

where $[N]$ is a shape function matrix; $\{d\}$ – displacement vector.

In the case of plane stress state and if the displacement vector is known, the deformation vector could be represented in the form of

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} = [B]\{d\}, \quad (4)$$

where $[B]$ – a matrix of the shape function derivatives.

Physical characteristics of the plate are taken into account by means of elasticity matrix $[D]$ with the size of 3 × 3 and are described through the Young's modulus *E* and Poisson's ratio *v*:

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix}, \quad (5)$$

where $D_x = \frac{E}{\left(1 - \nu^2\right)}$; $D_y = \frac{E}{\left(1 - \nu^2\right)}$; $D_{xy} = \frac{E}{2(1+\nu)}$ $D_1 = \frac{\nu\ E}{\left(1 - \nu^2\right)}$.

Then the stress vector will have the following form:

$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{d\}. \quad (6)$$

Stiffness matrix of finite element $[k]^e$ is given by

$$[k]^e = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix}. \quad (7)$$

For the isotropic plate of uniform thickness, on the basis of the variation principle and taking into account ratio of $\left[B_i\right]$ and $\left[B_j\right]$, the matrix will be given by [5]:

$$k_{ij} = t\int_A \left[B_i\right]^T \left[D\right]\left[B_j\right] dA\,, \tag{8}$$

where *i, j* are indices of the components of the displacements in the directions *x* and *y* respectively. After performing transformations (1) – (5) the stiffness matrix can be written in an explicit form as

$$[k]^e = \begin{bmatrix} \frac{3D_x I_1 + D_{xy} b^2 I_2}{3b} & -\frac{D_1 I_3 + D_{xy} I_4}{2} & -\frac{6D_x I_1 + D_{xy} b^2 I_2}{6b} & -\frac{D_1 I_3 - D_{xy} I_4}{2} \\ -\frac{D_1 I_5 + D_{xy} I_6}{2} & \frac{D_y b^2 I_7 + 3D_{xy} I_8}{3b} & \frac{D_1 I_5 - D_{xy} I_6}{2} & \frac{D_y b^2 I_7 + 6D_{xy} I_8}{6b} \\ -\frac{6D_x I_1 + D_{xy} b^2 I_2}{6b} & \frac{D_1 I_3 - D_{xy} I_4}{2} & \frac{3D_x I_1 + D_{xy} b^2 I_3}{3b} & \frac{D_1 I_3 - D_{xy} I_4}{2} \\ -\frac{D_1 I_5 - D_{xy} I_6}{2} & \frac{D_y b^2 I_7 - 6D_{xy} I_8}{6b} & \frac{D_1 I_5 + D_{xy} I_6}{2} & \frac{D_y b^2 I_7 + 3D_{xy} I_8}{3b} \end{bmatrix}, \tag{9}$$

where $I_1 = t\int_0^l Y_{mu} Y_{nu} dy$; $I_2 = t\int_0^l Y'_{mu} Y'_{nu} dy$; $I_3 = t\int_0^l Y_{mu} Y'_{nv} dy$; $I_4 = t\int_0^l Y'_{mu} Y_{nv} dy$; $I_5 = t\int_0^l Y'_{mv} Y_{nu} dy$; $I_6 = t\int_0^l Y_{mv} Y'_{nu} dy$; $I_7 = t\int_0^l Y'_{mv} Y'_{nv} dy$; $I_8 = t\int_0^l Y_{mv} Y_{nv} dy$.

Taking into account the orthogonality conditions, for the case of strip with hinged ends for $m \neq n$ we obtain:

$$\begin{array}{l} \int_0^l Y_m Y_n dy = 0 \\ \int_0^l Y''_m Y''_n dy = 0 \end{array}. \tag{10}$$

The vector of equivalent forces, concentrated in the nodes of finite strip $\{F\}^e$, has the following form [5]:

$$\{f\}^e = [N]^T \{q\}, \tag{11}$$

where $[q]$ is the vector of volumetric forces.

For the plane stress state, taking into account linear distribution of the displacement in the transverse direction, in the case of uniformly distributed load the vector of equivalent forces will be given by

$$\left\{f_q\right\}^e = \frac{b}{2}\begin{Bmatrix} q_x \int_0^l Y_m dy \\ q_y \frac{l}{\mu_m} \int_0^l Y'_m dy \\ q_x \int_0^l Y_m dy \\ q_y \frac{l}{\mu_m} \int_0^l Y'_m dy \end{Bmatrix}, \tag{12}$$

where q_x and q_y are the load vector components in *x, y* directions respectively.

On the basis of Lagrange variation principle of possible displacements, by the analogy with the finite element method [5], the main equation will have the form of

$$[K]\{d\} = \{F\} \quad [K] = \sum_e [k]^e \quad \{F\} = \sum_e \{f\}^e, \tag{13}$$

where $[K], \{F\}$ are stiffness matrix and the vector of equivalent loads obtained by means of assembling.

An example of the plane elasticity problem calculation by the finite strip method

Research object is a thin plate with the following dimensions: *t = 1mm*, *b = 10 mm* and *L = 200 mm.* Load $q = 0.01\ N / mm^2$ is distributed uniformly over the plate area. The plate is isotropic, Young's modulus $E = 206000\ N / mm^2$ and Poisson ratio $\nu = 0.3$.

For problem solution by the finite element method two-dimensional four-node elements CPLSTN4 from the library of software package Femap with NX Nastran are used [4]. The plate model with division into 40 elements in the transverse direction and into 800 elements in another direction is taken as a reference.

Table 1 presents maximal displacements of the console plate with the length *L = 200 mm* and divisions into 1, 3, 6 and 10 finite strips respectively as well as with different numbers of superimposed terms (half-waves).

Distribution of displacements along the plate length at the distances *L = 50, 100, 150* and *200mm* from the coordinate origin is presented in Fig. 2.

Table 1

Maximal displacements of the free end of the rectangular plate (L=200мм), $f_{\max}$

FSM					FEM	
The number of terms in the series r	$f^1_{\max}$	$f^3_{\max}$	$f^6_{\max}$	$f^{10}_{\max}$	1 element	40x800 elements
1	1.014	1.028	1.029	1.029	1.0627	1.061
3	1.060	1.076	1.078	1.078		
6	1.062	1.078	1.080	1.103		
10	1.063	1.078	1.080	1.107		

Table 2 presents a comparison of maximal stress σ_y distribution along the length of the plate for coordinates *L = 50, 100, 150, 200 mm* and *b = 0*. The stresses are obtained by FEM calculation of the reference model (40 x 800) as well as by using FSM (the number of terms in the series is 1, 3, 6 али 10 respectively). A relative error of the first strip calculation by FSM is given as compared with the reference model.

Table 2

Distribution of stress σ_y (MPa)

		L=50	L=100	L=150	L=200
FEM	σ_y	114.3	62.3	27.7	6.9
FSM (r=1)	σ_y	70.7	65.3	50.0	27.1
	%	38	5	81	293
FSM (r=3)	σ_y	103.7	72.7	26.7	9.1
	%	9	17	4	32
FSM (r=6)	σ_y	111.9	66.0	30.1	8.2
	%	2	6	9	19
FSM (r=10)	σ_y	114.2	67.3	29.4	6.9
	%	0	8	6	0

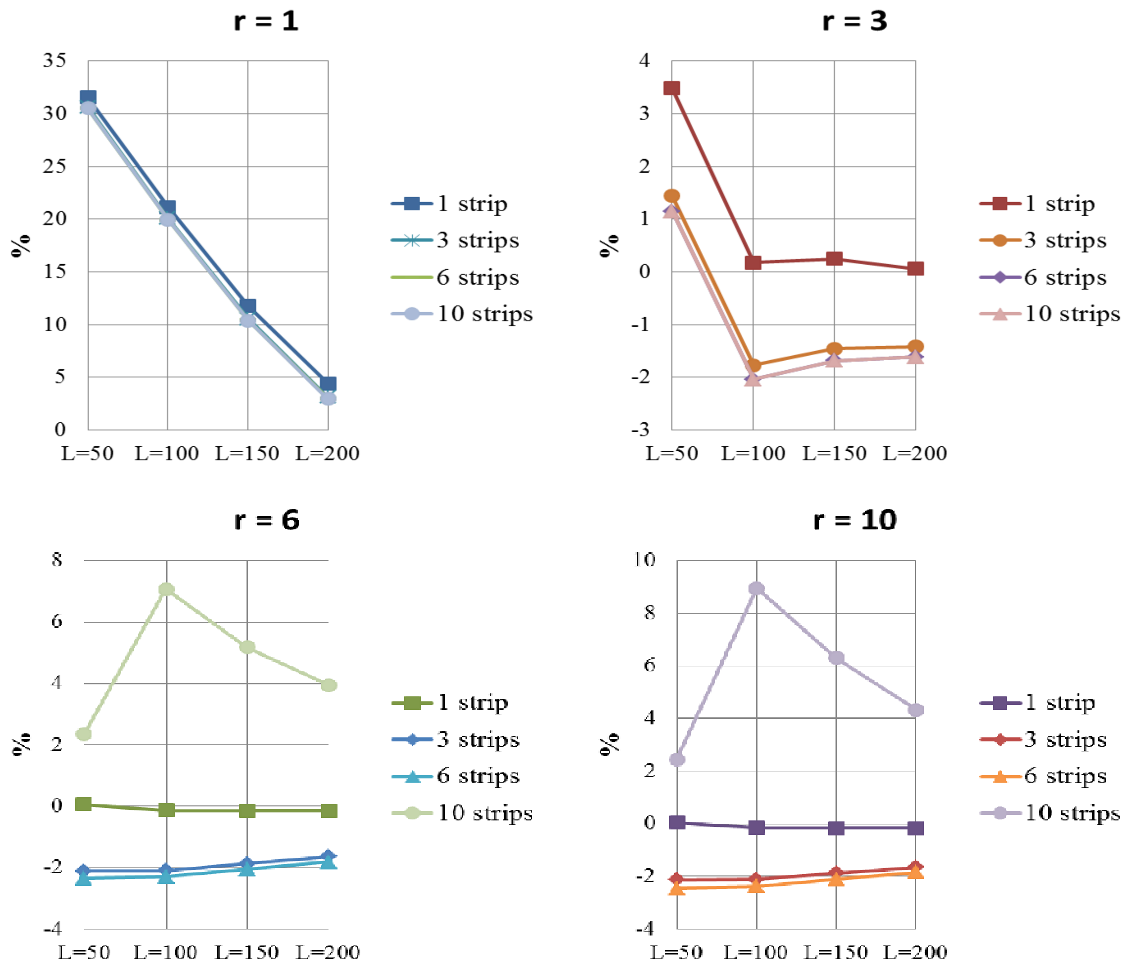

Fig. 2. Relative error of the displacement distribution along the plate length, when using FSM, in comparison with FEM

Conclusions

After analyzing numerical data, obtained as a result of calculation of the console plate by the finite strip method with simple boundary conditions, a conclusion can be made that this method makes it possible to obtain approximate solutions within the range of permissible errors. Under given simple boundary conditions, a single finite strip provides good description of the simulated plate.

Analysis of maximal displacements of the free end of the plate (Table 1) shows that an acceptable result $f^1 = 1.062$ can be obtained as compared with that, obtained by FEM - $f = 1.061$, simulating the plate by a single strip and summarizing only 6 terms of the series (Fig. 2). Stress distribution along the plate length has been obtained with maximal relative error 8 %, as compared with FEM, with the number of terms in the series r = 10.

It should be noted that FSM is most effective for calculating separate structures having constant physical-geometrical characteristics and simple boundary conditions in corresponding cross-sectional directions. Under these conditions the possibility of simplifying the design system of equations using a lower-order set of subsystems, each of which corresponds to its own component of the Fourier series in the longitudinal direction, is more convenient. By performing discretization of the problem in one direction only and using continuous trigonometric functions in the other direction, the process of solving the set problem could be simplified considerably. The above simplification reduces dimensionality of the problem and leads to significant simplifications in the design algorithm development. There is also corresponding reduction of the number of unknowns, the size and bandwidth of the coefficient matrix of the design system of linear algebraic equations are decreased and the amount of initial data is reduced. This feature of FSM determines its main advantage.

Among disadvantages of FSM we can mention the specificity of its usage: this method gives the best results while applied to the problems, the research objects of which have unchanged geometrical and physical characteristics along one of the coordinates, particularly, for calculating bridge, box-like and other elongated structures, especially plate structures.

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