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THE ANALYSIS OF ONE PROBLEM OF THE OPTIMUM APPROXIMATION ON THE LIMITING CYCLE

Bifurcational analysis has been carried out for one class of non-linear concentrated dynamic systems, the problem of optimum approximation on a limiting cycle is considered.

***Key words**: Bifurcation, attractor, bifurcational parameter, limiting cycle, optimum control.*

Introduction and problem definition. The description of mathematical model of various dynamic systems includes values, which are mathematically functions of time with complex nonlinear dependencies, and which are often almost impossible to identify. Therefore for the detailed analysis with the purpose of forecasting of such systems it is necessary to select the most significant variables from all the variety and then to examine dependencies only between them.

Thus, for the conditional closed economic systems particularly it is natural to select such basic variables: x - size of investment, y - profit received. Assuming, that the mode of dynamics is almost established (i.e. $t \to \infty$) and duration of capital circulation cycle $\tau << t$, dynamic variables $x(t)$ and $y(t+\tau) = y(t) + O(\tau) \approx y(t)$ can be correlated to one moment of time t and thereby to study dynamics in a phase plane. Further, it is quite natural to assume existence of functional connection $y = f\left(x, \frac{dx}{dt}\right)$ with some smooth function f. In fact, the profit received y depends on size of investment, but on the tendency of its reception (variation) in time also, i.e. on $\frac{dx}{dt}$. For the same reason, when considering a somewhat converse connection, we obtain that $x = g\left(y, \frac{dy}{dt}\right)$ with some smooth function g.

Let's assume that the price is managing parameter in considered economic system, to be exact in considered mathematical model. Identification of functions f and g for concrete considered economic system is a separate problem which partly depends on expert knowledge of structure of considered system. However, as a number of works show [4,7-8], wide enough class of problems is covered, when functions f and g are represented in the following additive form: $f(t_1,t_2) = H(t_1) + \sigma(t_2)$ и $g(t_1,t_2) = \varphi(t_1) + \theta(t_2)$. Further we will concretize a situation even more, assuming that $\sigma(t_2) = \sigma_{av} \cdot t$ and $\theta(t_2) = \theta_{av} \cdot t_2$. Then finally we come to following dynamic system:

$$\begin{cases} \dfrac{dx}{dt} = \alpha \cdot (H(x) - y), \\ \dfrac{dy}{dt} = \beta \cdot (x - \varphi(y)), \end{cases} \tag{1}$$

where $\sigma_{av} = -\frac{1}{\alpha}$ and $\theta_{av} = \frac{1}{\beta}$.

Note that functions H and φ , which take part in system creation (1), make obvious economic sense [8] and often are called characteristics.

Bifurcation analysis. Mathematical modeling of various problems of natural sciences often leads to studying of systems of the differential equations containing parameters. Such a studying usually begins by determination of fixed points (stationary decisions) in their dependence on parameters and detection of periodic modes as far as possible. Detection of birth-of-cycle bifurcation [1,3-4] (i.e. Hopf bifurcation) is one of stages of this research.

Below the problem of birth-of-cycle bifurcation origin in dynamical system is considered (1), when its characteristics obviously depend on the managing parameter of the price p:

$$\begin{cases} \frac{dx}{dt} = \alpha \cdot (H(x,p) - y), \\ \frac{dy}{dt} = \beta \cdot (x - \varphi(y,p)), \end{cases} \tag{2}$$

where $\alpha, \beta = const > 0$ and p - bifurcation parameter.

Concerning system (2) we assume the following: $\begin{cases} H(\xi, p) = y_o \\ \varphi(y_o, p) = \xi \end{cases}$ with any p of some interval I_{p_o}, (ξ, y_o)- isolated fixed system point, functions $H(x,p)$ and $\varphi(y,p)$ are analytical on sets $I_\xi \times I_{p_o}$ and $I_{y_o} \times I_{p_o}$ accordingly. So we have

THEOREM. Let 1) $\alpha \frac{\partial H(\xi, p_o)}{\partial x} = \beta \frac{\partial \varphi(y_o, p_o)}{\partial y}$;

2) $\alpha \frac{\partial^2 H(\xi, p_o)}{\partial x \partial p} \neq \beta \frac{\partial^2 \varphi(y_o, p_o)}{\partial y \partial p}$;

3) $\left| \frac{\partial H(\xi, p_o)}{\partial x} \right| < \sqrt{\frac{\beta}{\alpha}}$ или $\left| \frac{\partial \varphi(y_o, p_o)}{\partial y} \right| < \sqrt{\frac{\alpha}{\beta}}$.

Then $\exists$ number $\varepsilon_{Hopf} > 0$ and analytical function $\mu_{Hopf}(\varepsilon)$, $0 < \varepsilon < \varepsilon_{Hopf}$, such, that $\forall\ \varepsilon \in (0, \varepsilon_{Hopf})$ system (2) with $p = p_{Hopf}(\varepsilon)$ has analytical periodic solution $[x_\varepsilon(t), y_\varepsilon(t)]$, with period $T_{Hopf}(\varepsilon) = \frac{2\pi}{\omega_o}\left\{1 + \sum_{j=2}^{\infty} \tau_j \varepsilon^j \right\}$, where $\omega_o = \sqrt{\alpha\beta - \frac{1}{4}\left(\alpha \frac{\partial H(\xi, p_o)}{\partial x} + \beta \frac{\partial \varphi(y_o, p_o)}{\partial y}\right)^2}$.
Moreover, for every $L > \frac{2\pi}{\omega_o}$ there is a neighborhood of $\Re$ point(ξ, y_o) and interval $\tilde{I} \ni p_o$, such that for every price value $p \in \tilde{I}$ the only nonconstant periodic solutions of system (2) laying in $\Re$ and having the period smaller than L, are elements of a set $[x_\varepsilon(t), y_\varepsilon(t)]$, where $p_{Hopf}(\varepsilon) = p$ and $\varepsilon \in (0, \varepsilon_{Hopf})$.

THE PROOF. Let $\Omega = \begin{bmatrix} \alpha \cdot (H(x,p) - y), \\ \beta \cdot (x - \varphi(y,p)) \end{bmatrix}$, then Jacobi Matrix of this vector looks like: $\frac{\partial \Omega}{\partial(x,y)} = \begin{bmatrix} \alpha \frac{\partial H(\xi, p_o)}{\partial x} & -\alpha \\ \beta & -\beta \frac{\partial \varphi(y_o, p_o)}{\partial y} \end{bmatrix}$.

Let $\lambda(p)$ and $\bar{\lambda}(p)$ be roots of the characteristic equation:

$$\det\left[\frac{\partial \Omega}{\partial(x,y)} - \lambda E\right] = 0,$$

Then we can elementary confirm, that:

$$\mathrm{Re}\lambda(p) = \frac{1}{2}\left(\alpha \frac{\partial H}{\partial x} - \beta \frac{\partial \varphi}{\partial y}\right) \text{ and } \mathrm{Im}\lambda(p) = \sqrt{\alpha\beta - \frac{1}{4}\left(\alpha \frac{\partial H}{\partial x} + \beta \frac{\partial \varphi}{\partial y}\right)^2}.$$

Further from Hopf theorem written down in the form from [1], follows, that for the proof of our theorem it is enough to show, that:

$$\operatorname{Re}\lambda(p_o)=0\,,\ \frac{d}{dp}\operatorname{Re}\lambda(p_o)\neq 0 \ \text{ and } \ \omega_o \overset{def}{=} \operatorname{Re}\lambda(p_o)>0\,.$$

But from 1) automatically follows, that $\operatorname{Re}\lambda(p_o)=0$, and as

$$\frac{d}{dp}\operatorname{Re}\lambda(p_o)=\frac{1}{2}\left(\alpha\frac{\partial^2 H(\xi,p_o)}{\partial x\partial p}-\beta\frac{\partial^2\varphi(y_o,p_o)}{\partial y\partial p}\right),$$

then from 2) condition follows, that $\frac{d}{dp}\operatorname{Re}\lambda(p_o)\neq 0$.

Now from 1) and 3) conditions follows, that $\operatorname{Im}\lambda(p_o)>0$. In fact, let, for example $\left|\frac{\partial H(\xi,p_o)}{\partial x}\right|<\sqrt{\frac{\beta}{\alpha}}$, then we have: $\{\operatorname{Im}\lambda(p_o)\}^2=\alpha\beta-\frac{1}{4}\left(\alpha\frac{\partial H}{\partial x}+\beta\frac{\partial\varphi}{\partial y}\right)^2=$

$=\alpha\beta-\alpha^2\left(\frac{\partial H}{\partial x}\right)^2>\alpha\beta-\alpha^2\frac{\beta}{\alpha}=0$, as was to be shown. So, the theorem is proved.

Further, as [4,7] works prove, we will use following approximations (fig.1), approaching function $H(x,p)$ by a polynomial of 3-rd degree, and also assuming $\varphi(y,p)=k\cdot\sqrt{y}$. In this connection following consequence of the above proved theorem is required.

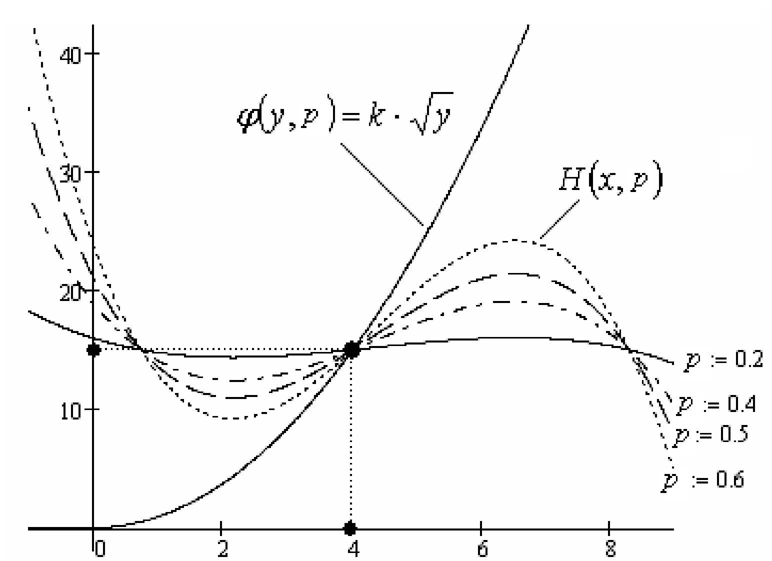

Fig. 1. The illustration of H and φ characteristics deformation when changing price p as bifurcation parameter

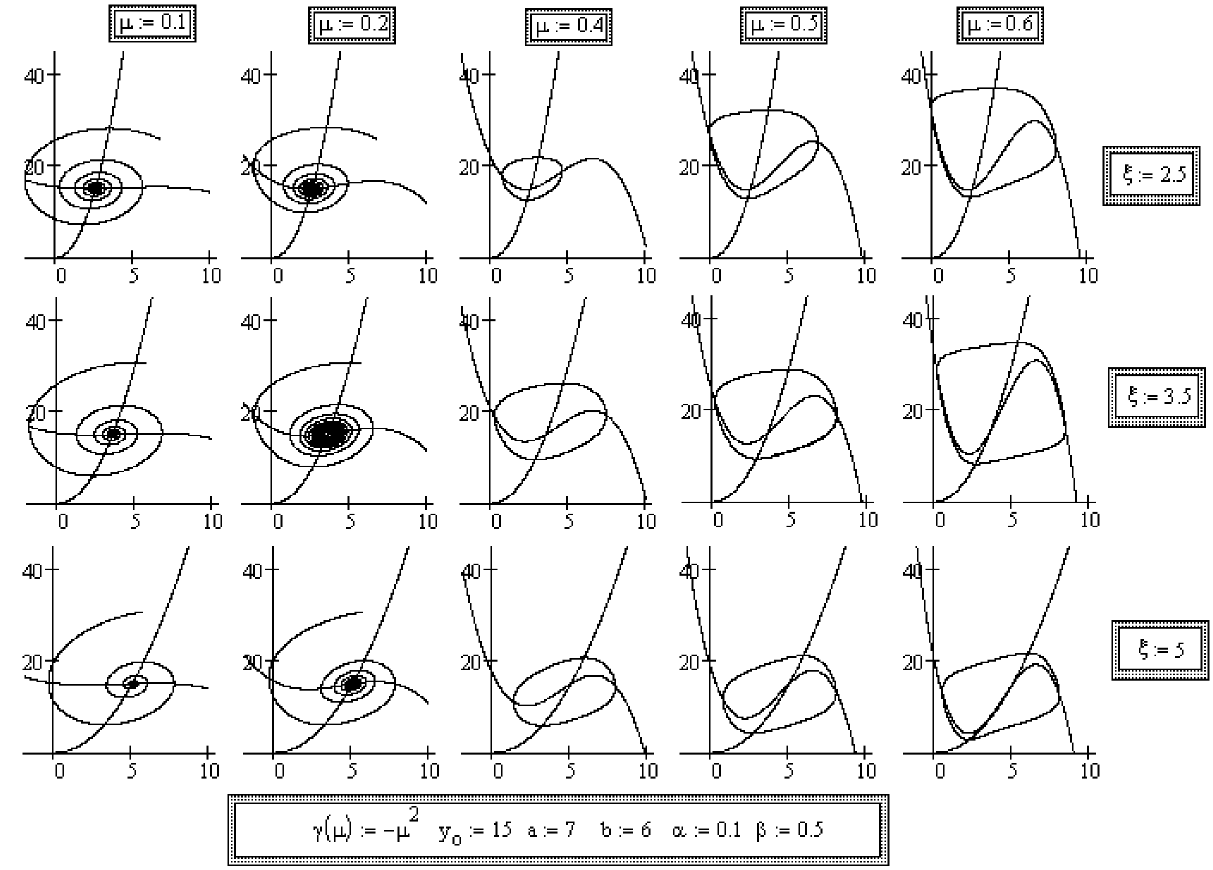

Fig. 2. The Illustration of Hopf bifurcation in phase space of system (2)

THE COROLLARY. Let $0 < b < a$, $\Psi_{a,b}(x) = x \cdot (x-a)(x-b)$, $h_p(x) = \gamma(p) \cdot \Psi_{a,b}(x)$, $H(x,p) = h_p(x) - h_p(\xi) + y_o$, $\varphi(y,p) = \xi \cdot \sqrt{\frac{y}{y_o}}$, $\xi \notin \{0,a,b\}$, $\frac{d\gamma(p_o)}{dp} \neq 0$, $\gamma(\mu_o) \cdot \Psi_{a,b}(\xi) < 2\frac{y_o}{\xi}$.

Then if $\frac{\alpha}{\beta} = \frac{1}{2}\frac{\xi}{y_o}\gamma(p_o) \cdot \Psi_{a,b}(\xi)$, so $\exists\ p \approx p_o$ by which system (2) has periodic solution.

Nature of reorganization of a phase system (2) portrait at a variation of bifurcation parameter p, and equilibrium values of the investment ξ also is presented in Figure 2. Corresponding dynamics in the expanded phase space is presented in FIG 3.

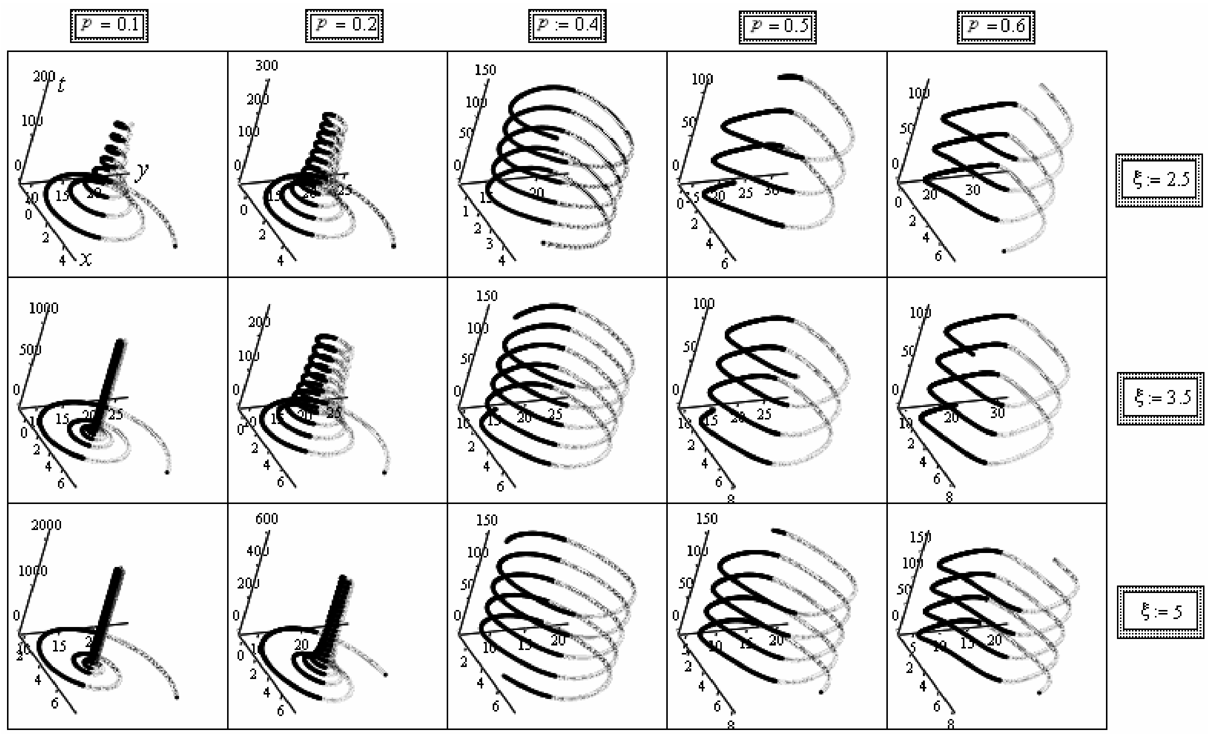

Fig. 3. The Illustration of Hopf bifurcation in expanded phase space

The analysis of management problem. As is known [1,3-4], attractors of dissipative dynamical systems describe their dynamics in the established mode, i.e. any phase trajectory beginning in the attraction pool eventually comes nearer to it unrestrictedly, or as they say, leads to attractor. In practice all systems are dissipative. The stability of dynamics is important from economic point of view, and all transients are only the intermediate stages. In this connection the problem of minimization of time of getting the attractor (i.e. time of a transitive mode) is actual at the preset entry conditions.

Thus, as a rule, values connected with the systems resources act as managing parameters. In our case as the managing parameter the price p is considered, and as a quality functional minimization of time of getting the limit cycle of system (2) at the preset entry conditions. Set of admissible managements U^{∂} is defined as follows:

$$U^{\partial} = \{p \in KC[t_0, t_1] : p_{\min} \le p(t) \le p_{\max} \ \ \forall t \in [t_0, t_1]\}, \quad (3)$$

where by $KC[t_0, t_1]$ the set of all piece-continuous functions on a piece $[t_0, t_1]$ is designated, and the ends of a piece are not fixed.

As follows from the standard Pontriagin's maximum principle [2] optimal managements are piece-constant functions. Therefore, having entered into consideration a class of managements:

$$U^{\partial}[p_0, \varepsilon, T, \varphi_0] = \{p(t) = p_0 + \varepsilon \operatorname{sgn}[\sin(T \cdot t + \varphi_0)]\} \subset U^{\partial}, \quad (4)$$

we get that for optimal management $\tilde{p}(t)$ there will be such values $p_0, \varepsilon, T, \varphi_0$, that inclusion $\tilde{p}(t) \in U^{\partial}[p_0, \varepsilon, T, \varphi_0]$ is valid.

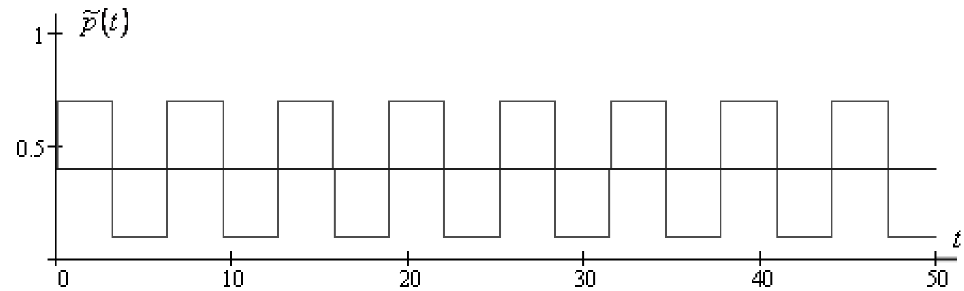

Fig. 4. Dependence of optimum management with $p_o = 0.4$, $\varepsilon = 0.3$, $T = 1$, $\varphi_0 = 0$

The result of numerical integration of system of the equations (2) at optimum strategy of the price $\tilde{p}(t)$ (fig.4) is represented in Figure 5.

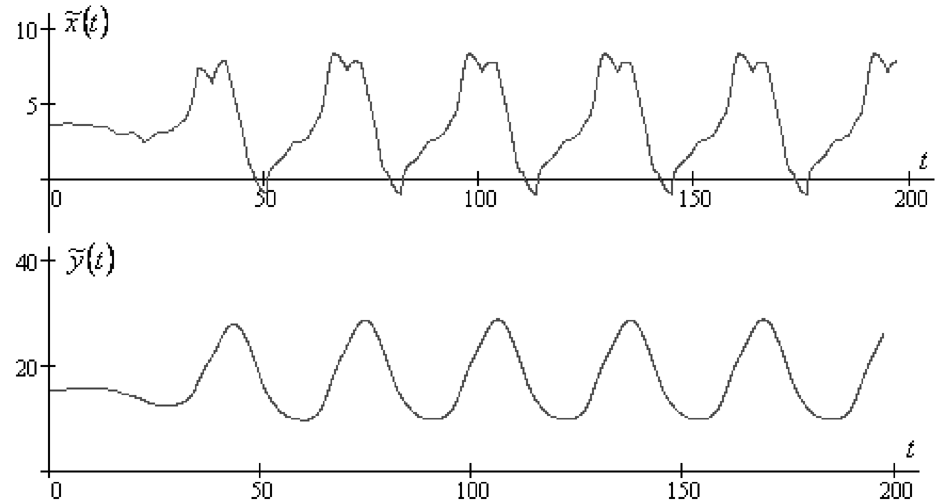

Fig. 5. Time dependence of optimum trajectories $\tilde{x}(t)$ and $\tilde{y}(t)$

Conclusion. In the framework of mathematical model considered the accomplished analysis shows that at optimal price management strategy, functions of investments and profits oscillate synchronously with equal frequencies, but nature of investments less smooth and has splashes around a maximum zone. It is interesting that at optimum strategy of profit function there are segments with negative capital which for example it is possible to interpret as additional profit on corresponding strategy of investments.

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